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GREEN'S FUNCTION OF THE STATIONARY DYNAMIC PROBLEM FOR A VISCOELASTIC HALF-SPACE*

D.R. ATADZHANOV, A.G. SARKISYAN and A.I. TSEITLIN

A stationary Green's function is constructed for a viscoelastic half-plane in the form of a Fourier integral, and asymptotic estimates obtained for the integrand are used to carry out a numerical realization efficiently.

Green's function was constructed for an elastic half-plane in /1/, but was complicated and difficult to calculate. A simpler expression was given for this function in /2/ with reference to a source which was not easily accessible. A solution of the problem of the steady-state oscillations of a homogeneous, isotropic viscoelastic half-plane, caused by the action of an arbitrarily oriented concentrated force varying harmonically with time, is given below. The solution is used as the basic for constructing the BEMDYST boundary-element program.

1. We shall consider the half-plane $P\ (-\infty < x_1 < \infty,\ x_2 \geqslant 0)$ in Cartesian coordinates x_1, x_2. We will assume that the material of the medium possesses internal friction, and we will describe it using the model of relative, frequency-independent damping /3/. The complex modulus is taken in the form

$$E(\omega) = E_1 \mid \omega \mid^{\alpha} (\cos {}^1/_2\, \pi\alpha + i \operatorname{sign} \omega \sin {}^1/_2 \pi\alpha) \tag{1.1}$$
$$E_1 = \mid E(1) \mid, \quad \alpha = 2\pi^{-1} \operatorname{arctg} \gamma$$

and Poisson's ratio ν is assumed to be a real constant, i.e. it is assumed to be non-relaxing. Here γ is the loss factor. The model ensures that energy losses on the whole frequency axis are frequency independent, and enables us to study any dynamic processes. It cannot however deal with a static load caused by unlimited creep when $\omega = 0$.

Let a concentrated unit force varying harmonically with time be applied to the point $a\,(a_1, a_2)$. Eliminating the time coordinate, we reduce the problem to that of determining the complex Green's matrix function $g\,(x; a \mid \omega) = [g_{kj}\,(x; a \mid \omega)]_{2\times 2}$, satisfying the Lamé equation and boundary conditions

$$A\,(\partial_x; i\omega)\ g\,(x; a \mid \omega) + \rho\omega^2 g\,(x; a \mid \omega) + \delta\,(x, a)\, I = 0$$
$$\lim_{(P\setminus\Gamma)\ni x\to x_0\in\Gamma} T\,(\partial_x, \mathbf{n}\,(x_0); i\omega)\ g\,(x; a \mid \omega) = 0$$

Here $g_{kj}\,(x; a \mid \omega)$ is the displacement at the point $x\,(x_1, x_2)$ along the x_k axis caused by unit force acting in the direction x_j at the point $a\,(a_1, a_2)$; $A\,(\partial_x;\ i\omega) = [A_{kj}\,(\partial_x;\ i\omega)]_{2\times 2}$, $T\,(\partial_x, \mathbf{n}\,(x_0);\ i\omega) = [T_{kj}\,(\partial_x,\ \mathbf{n}\,(x_0);\ i\omega)]_{2\times 2}$ are the complex differential matrix operators obtained, respectively, from the static matrix Lamé operator and stress operator of the classical theory of elasticity, where the elastic constants have been replaced by the corresponding complex moduli of viscoelasticity

$$A_{kj}\,(\partial_x; i\omega) = \delta_{kj}\mu\Delta + (\lambda + \mu)\, \partial^2/\partial x_k \partial x_j$$
$$T_{kj}\,(\partial_x, \mathbf{n}\,(x); i\omega) = \lambda n_k\,(x)\, \partial/\partial x_j + \mu n_j\,(x)\, \partial/\partial x_k + \mu\delta_{kj} \partial/\partial \mathbf{n}\,(x)$$
$$\lambda = \nu E\,(\omega)/((1 + \nu)\,(1 - 2\nu)), \quad \mu = E\,(\omega)/(2\,(1 + \nu))$$

δ_{kj} is the Kronecker delta, $\delta\,(x, a) = \delta\,(x_1 - a_1)\,\delta\,(x_2 - a_2)$ is the Dirac delta function, I is the unit matrix, ρ is the density of the material of the half-plane, λ, μ are complex parameters representing the analogues of the corresponding Lamé constants, $E\,(\omega)$ is obtained from (1.1), Γ is the boundary of the half-plane, $\mathbf{n}\,(x_0) = \{0, 1\}$ is the vector of inner normal to the boundary Γ, Δ is the Laplace operator.

We can write /4/ the regular solution of the problem in question, taking into account the decay of the oscillations at infinity, in the form

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$$c(x^*)\,g(x^*;a\,|\,\omega) = G(x^*;a\,|\,\omega) + \int G_q(x_0;x^*\,|\,\omega)\,g(x_0;a\,|\,\omega)\,dx_1 \tag{1.2}$$
$$c(x^*) = \{1 \quad \text{for} \quad x^* \in (P \setminus \Gamma);\ {}^1/_2 \quad \text{for} \quad x^* \in \Gamma\}$$
$$G_q(x_0;x^*\,|\,\omega) = [T(\partial_x, \mathbf{n}(x_0);\ i\omega)\,G(x;x^*)\,|\,\omega)]^T\,|_{x_2=0}$$

where $G(x^*;x\,|\,\omega)$ is the stationary Green's function for the viscoelastic plane. Here and henceforth the integration in x_1, x_1^* and ξ_2 will be carried out from $-\infty$ to $+\infty$. By virtue of the relation

$$G_p(x_0^*;x_0\,|\,\omega) \equiv [T(\partial_{x^*}, \mathbf{n}(x_0);\ i\omega)\,G(x^*;x_0\,|\,\omega)]^T\,|_{x_2^*=0} = -G_q(x_0;x_0^*\,|\,\omega) \tag{1.3}$$

the solution (1.2) for $x^* \in \Gamma$ can be written more conveniently in the form

$$^1/_2 g(x_0^*;a\,|\,\omega) = G(x_0^*;a\,|\,\omega) - \int G_p(x_0^*;x_0\,|\,\omega)\,g(x_0;a\,|\,\omega)\,dx_1 \tag{1.4}$$

and the integral is understood to represent its principal value.

Let us apply to both sides of Eq.(1.4) the Fourier transform in x_1^* /5/

$$^1/_2 g_F(\xi_1,0;a\,|\,\omega) = G_F(\xi_1,0;a\,|\,\omega) - \tag{1.5}$$
$$\iint G_p(x_1^*,0;x_0\,|\,\omega)\,g(x_0;a\,|\,\omega)\exp(i\xi_1 x_1^*)\,dx_1^*\,dx_1$$

where the subscript F denote the Fourier transform. Since $G_p(x_0^*;x_0\,|\,\omega)$ is a function of the difference $x_0^*-x_0$, it follows that the double integral in (1.5) can be written in the form

$$G_{pF}^e(\xi_1\,|\,\omega)\ g_F(\xi_1,0;a\,|\,\omega)$$
$$G_{pF}^e(\xi_1\,|\,\omega) = \int G_p(R\,|\,\omega)\exp(i\xi_1 R)\,dR \quad (R = x_1^* - x_1)$$

Taking all this into account, we reduce relation (1.5) to the form

$$[{}^1/_2 I + G_{pF}^e(\xi_1\,|\,\omega)]\ g_F(\xi_1,0;\ a\,|\,\omega) = G_F(\xi_1,0;a\,|\,\omega) \tag{1.6}$$

and this yields

$$g_F(\xi_1,0;\ a\,|\,\omega) = [C]^{-1}\,G_F(\xi_1,0;\ a\,|\,\omega), \quad C = {}^1/_2 I + G_{pF}^e(\xi_1\,|\,\omega) \tag{1.7}$$

The use in (1.7) of the stationary Green's function G for the viscoelastic plane and the stress tensor G_p generated by it, leads to very bulky and time-consuming calculations. This can be overcome as follows. We have

$$G_F(\xi_1,0;a\,|\,\omega) = (2\pi)^{-1}\int G_{FF}(\xi_1,\xi_2;a\,|\,\omega)\exp(-i\xi_2 x_2^*)\,d\xi_2\,|_{x_2^*=0} \tag{1.8}$$
$$G_{pF}^e(\xi_1,x_2^*\,|\,\omega) = (2\pi)^{-1}\exp(-i\xi_1 x_1)\int G_{pFF}(\xi_1,\xi_2;x_0\,|\,\omega)\exp(-i\xi_2 x_2^*)\,d\xi_2 \tag{1.9}$$

where G_{FF}, G_{pFF} are double Fourier transform of G and G_p. Applying in the Lamé equations the double Fourier transformation in x^*, we find

$$G_{FFmj}(\xi_1,\xi_2;x\,|\,\omega) = \frac{\exp[i(\xi_1 x_1+\xi_2 x_2)]}{\rho\omega^2}\left(\frac{\xi_m\xi_j}{\xi^2+k_l^2} + \frac{\delta_{mj}k_t^2-\xi_m\xi_j}{\xi^2-k_t^2}\right) \tag{1.10}$$
$$k_l^2 = \rho\omega^2/(\lambda+2\mu), \quad k_t^2 = \rho\omega^2/\mu, \quad \xi^2 = \xi_1^2+\xi_2^2$$

In order to determine $G_{pFF}(\xi_1,\xi_2;x_0\,|\,\omega)$, we can also apply the double Fourier transformation in x^* to the relation

$$G_p(x^*;x_0\,|\,\omega) = [T(\partial_{x^*}, \mathbf{n}(x^*);\ i\omega)\,G(x^*;x_0\,|\,\omega)]^T$$

and integrate by parts on the right-hand side

$$G_{pFFjk}(\xi_1,\xi_2;x_0\,|\,\omega) = -in_l(x^*)(\lambda\delta_{lk}\xi_m + \mu\xi_k\delta_{lm} + \mu\xi_l\delta_{km})\,G_{FFmj}(\xi_1,\xi_2;x_0\,|\,\omega) \tag{1.11}$$

Substituting (1.11) into (1.9) and taking into account (1.10), integrating /6/

$$2\pi i\rho\omega^2 G_{pFjk}^e(\xi_1,x_2^*\,|\,\omega) = \lambda\xi_1^2\delta_{2k}((\xi_j-\delta_{2j}\xi_2)I_0+\delta_{2j}I_1) + \tag{1.12}$$
$$\lambda\delta_{2k}((\xi_j-\delta_{2j}\xi_2)I_2+\delta_{2j}I_3) + \lambda k_t^2\delta_{2k}((\xi_j-\delta_{2j}\xi_2)I_{0t}+\delta_{2j}I_{1t}) +$$
$$2\mu((\xi_k-\delta_{2k}\xi_2)(\xi_j-\delta_{2j}\xi_2)I_1 + (\xi_j-\delta_{2j}\xi_2)\delta_{2k}I_2 + (\xi_k-\delta_{2k}\xi_2)\delta_2 I_2 +$$
$$\delta_{2k}\delta_{2j}I_3) + \mu k_t^2(\delta_{2j}(\xi_k-\delta_{2k}\xi_2)I_{0t} + \delta_{2j}\delta_{2k}I_{1t} + \delta_{kj}I_{1t})$$

where

$$I_n = I_{nl} - I_{nt}, \quad I_{ns} = (-i)^n \pi R_s^{n-1} \exp(-R_s x_2^*)$$
$$R_s = \operatorname{sign}\omega \sqrt{\xi_1^2 - k_s^2}, \quad n = 0, 1, 2, 3, \quad s = l, t$$

and passing to the limit as $(P \setminus \Gamma) \ni x^* \to x_0^* \in \Gamma$ (written otherwise as $x_2^* \to +0$), we obtain

$$2i\rho\omega^2 G_{pFjk}^e(\xi_1, +0 \mid \omega) = (\lambda + 2\mu)\,\delta_{2k}(\xi_j - \delta_{2j}\xi_2)(R_t - R_l) +$$
$$\lambda\xi_1^2\delta_{2k}(\xi_j - \delta_{2j}\xi_2)(R_l^{-1} - R_t^{-1}) + \lambda k_t^2\delta_{2k}(\xi_j - \delta_{2j}\xi_2) R_t^{-1} +$$
$$2\mu\delta_{2j}(\xi_k - \delta_{2k}\xi_2)(R_t - R_l) + \mu k_t^2(\delta_{2j}(\xi_k - \delta_{2k}\xi_2) R_t^{-1} - i\delta_{kj})$$

Taking into account the relations between the boundary and straight (singular) values of the generalized double layer potential /4/

$$G_{pF}^e(\xi_1 \mid \omega) = {}^1/_2 I + G_{pF}^e(\xi_1, +0 \mid \omega)$$

we find, after the transformation,

$$G_{pFjk}^e(\xi_1 \mid \omega) = {}^1/_2(\delta_{2k}(\xi_j - \delta_{2j}\xi_2) R_l^{-1} - \delta_{2j}(\xi_k - \delta_{2k}\xi_2) R_t^{-1})\,\alpha(\xi_1)$$
$$\alpha(\xi_1) = ik_t^{-2}(2(R_t - R_l) R_t + k_t^2)$$

Further, according to the second equation of (1.7),

$$C = {}^1/_2 \begin{Vmatrix} 1 & \xi_1 R_l^{-1}\alpha(\xi_1) \\ -\xi_1 R_t^{-1}\alpha(\xi_1) & 1 \end{Vmatrix}$$

Now, using the first relation of (1.7) (taking into account (1.8) and (1.10)) and (1.12) (taking into account (1.3)), we finally obtain Green's matrix

$$g(x^*, a \mid \omega) = G(x^*, a \mid \omega) - \frac{1}{2\pi}\int_{-\infty}^{\infty} B(\xi_1, x_2^*; a_2 \mid \omega)\times \tag{1.13}$$
$$\exp[-i\xi_1(x_1^* - a_1)]\, d\xi_1 \quad (x^*, a \in P)$$
$$B_{11} = \frac{\xi_1^2}{\Lambda_-(\xi_1)\,\mu k_t^2 R_l}\Big(N_+(\xi_1)\Big(\frac{R_t R_l}{2\xi_1^2}\exp[-R_t(x_2^* + a_2)] +$$
$$\frac{1}{2}\exp[-R_l(x_2^* + a_2)]\Big) - 2(2\xi_1^2 - k_t^2) R_l R_t(\exp[-(R_t x_2^* + R_l a_2)] +$$
$$\exp[-(R_l x_2^* + R_t a_2)]\Big)\Big)$$

$$B_{12} = \frac{i\xi_1}{\Lambda_-(\xi_1)\,\mu k_t^2}\Big(\frac{1}{2}N_+(\xi_1)(\exp[-R_t(x_2^* + a_2)] + \exp[-R_l(x_2^* + a_2)]) -$$
$$2(2\xi_1^2 - k_t^2) R_l R_t \exp[-(R_t x_2^* + R_l a_2)] -$$
$$2\xi_1^2(2\xi_1^2 - k_t^2)\exp[-(R_l x_2^* + R_t a_2)]\Big)$$
$$N_\pm(\xi_1) = (2\xi_1^2 - k_t^2)^2 \pm 4\xi_1^2 R_l R_t$$

The quantity B_{22} is obtained from B_{11} be replacing R_t by R_l and R_l by R_t, and the quantity B_{21} is obtained from B_{12} using the same substitution and change of sign, and $N_-(\xi_1)$ is a Rayleigh function.

To simplify the notation, we shall henceforth omit the asterisks. The accuracy of the solution obtained matches, for $a_2 = 0$ and elastic constants, that of the solution of Lamb's problem /7/ for an elastic half-plane. Comparison with Green's function given in /2/ and mentioned at the beginning of this paper, revealed an error in the latter.

2. The problems encountered in calculating the term appearing in (1.13) outside the integral sign, i.e. of Green's function for a viscoelastic plane, were discussed earlier by Sarkisyan* (*Sarkisyan A.G., Use of the method of boundary elements in solving plane, stationary dynamic problems of viscoelasticity. Candidate Dissertation, Moscow, 1986.). We shall now discuss methods of evaluating the integral term, i.e. the Fourier transformation for the elements of the matrix $B(\xi_1, x_2; a_2 \mid \omega)$. The latter are even functions of ξ_1 when $j = k$, and odd function of ξ_1 when $j \neq k$.

Taking these relations into account, we shall write the integral term in (1.13) in the form

$$\pi^{-1}\delta_{jk}I_c(0,\infty) - i\pi^{-1}(1-\delta_{jk})I_s(0,\infty)$$

$$I_c(p,q) = \int_p^q B_{jj}(\xi_1, x_2; a_2 \mid \omega)\cos[\xi_1(x_1 - a_1)]\,d\xi_1$$

$$I_s(p,q) = \int_p^q B_{jk}(\xi_1, x_2; a_2 \mid \omega)\sin[\xi_1(x_1 - a_1)]\,d\xi_1$$

We shall also separate the integration intervals in $I_c(0,\infty)$ and $I_s(0,\infty)$ into finite and semi-infinite intervals with the points $\xi_1 = b$ and $\xi_1 = d$, respectively.

Since $\cos[\xi_1(x_1 - a_1)]$ and $\sin[\xi_1(x_1 - a_1)]$ oscillate rapidly for large values of $x_1 - a_1$, it follows that in order to obtain the value of the integral with the necessary accuracy, we must take in the corresponding quadrature formula the polynomials of degree $n \gg n_*$, where l is the length of the interval of integration and $n_* = (x_1 - a_1)\,l/\pi + 1$ is the number of zeros in the integrand. This obviously requires very bulky computations. A quadrature formula useful for a wide range of variation in the frequency of the oscillating function was given in /8/. According to this formula we obtain, for the integrals over the finite intervals,

$$I_c(0,b) \approx J\sum_{m=0}^{n_1-1} B_{jj}(\xi_1^m, x_2; a_2 \mid \omega)\cos\left[\frac{1}{2}(x_1 - a_1)(\xi_1^{m+1} + \xi_1^m)\right]$$

$$I_s(0,d) \approx J\sum_{m=0}^{n_2-1} B_{jk}(\xi_1^m, x_2; a_2 \mid \omega)\sin\left[\frac{1}{2}(x_1 - a_1)(\xi_1^{m+1} + \xi_1^m)\right]$$

$$J = [{}^1/_2(x_1 - a_1)\Delta\xi_1]^{-1}\sin[{}^1/_2(x_1 - a_1)\Delta\xi_1]\,\Delta\xi_1$$

$$\xi_1^0 = 0,\quad \xi_1^1 = \Delta\xi_1, \ldots, \xi_1^{n_1} = n_1\Delta\xi_1 = b\,(\xi_1^{n_2} = n_2\Delta\xi_1 = d);\quad \Delta\xi_1 = \xi_1^{m+1} - \xi_1^m$$

Next we shall evaluate the integrals over semi-infinite intervals. After simple but lengthy reduction we obtain the following asymptotic relations as $\xi_1 \to \infty$, for the elements of the matrix $B(\xi_1, x_2; a_2 \mid \omega)$:

$$B_{jk}(\xi_1, x_2; a_2 \mid \omega) \sim -D_{jk}\mu^{-1}\xi_1^{-1}\exp[-\xi_1(x_2 + a_2)] \tag{2.1}$$

$$D_{11} = D_{22} = [4(1-\nu)^2 + (1-2\nu)^2]/[8(1-\nu)] \tag{2.2}$$

$$D_{12} = -D_{21} = -i\,({}^1/_2 - \nu) \tag{2.3}$$

Taking into account (2.1) and (2.2), we obtain the estimate

$$|\operatorname{Re}(I_c(b,\infty))| \sim$$

$$\left|-\operatorname{Re}(\mu^{-1})D_{11}\int_b^\infty \xi_1^{-1}\exp[-\xi_1(x_2 + a_2)]\cos[\xi_1(x_1 - a_1)]\,d\xi_1\right| \leqslant$$

$$\operatorname{Re}(\mu^{-1})D_{11}\int_b^\infty \exp[-\xi_1(x_2 + a_2)]\,d\xi_1 =$$

$$\operatorname{Re}(\mu^{-1})D_{11}(x_2 + a_2)^{-1}\exp[-b(x_2 + a_2)] \quad (b \gg 1)$$

Using analogous reasoning we obtain, by virtue of (2.1) and (2.3), $|\operatorname{Re}(I_s(d,\infty))| \leqslant |\operatorname{Re}(i\mu^{-1})|\,({}^1/_2 - \nu)(x_2 + a_2)^{-1}\exp[-d(x_2 + a_2)]\quad (d \gg 1)$

When $x_2 = a_2 = 0$, we obtain the estimates for the integrals in question, taking into account the asymptotic behaviour of the integral cosine and integral sine /9/, as follows:

$$\operatorname{Re}(I_c(b,\infty)) \approx \operatorname{Re}(\mu^{-1})D_{11}[b(x_1 - a_1)]^{-1}\sin[b(x_1 - a_1)]$$

$$\operatorname{Re}(I_s(d,\infty)) \approx \pm\operatorname{Re}(i\mu^{-1})({}^1/_2 - \nu)[d(x_1 - a_1)]^{-1}\cos[d(x_1 - a_1)]$$

$$(b \gg 1/|x_1 - a_1|,\ d \gg 1/|x_1 - a_1|,\ x_1 - a_1 \neq 0)$$

The upper sign corresponds to the integral with integrand B_{12}, and the lower sign to the integral with B_{21}.

We obtain analogous estimates for $\operatorname{Im}(I_c(b,\infty))$ and $\operatorname{Im}(I_s(d,\infty))$ by replacing, respectively, $\operatorname{Re}(\mu^{-1})$ by $\operatorname{Im}(\mu^{-1})$ and $\operatorname{Re}(i\mu^{-1})$ by $\operatorname{Im}(i\mu^{-1})$.

The solution obtained is most effective for the far field, i.e. when the displacements are determined far from the point of application of the concentrated harmonic force. The computing time increases substantially when the force is situated near the origin of coordinates and the displacements are determined in the same region.

The figure shows the results of solving (1.13) numerically on a computer, at $x_2 = 0$ (at the boundary of the half-plane), with the source situated at a depth of $a_2 = 30$ m, at an angular frequency $\omega = 62.28$ radians/sec, a modulus of elasticity $E_1 = 2.2$ GPa, a density $\rho = 2.2$ tons/m^3, a loss factor $\gamma = 0.2$ and a Poisson's ratio $\nu = 0.15$. The numbers on the curves correspond to the indices j, k

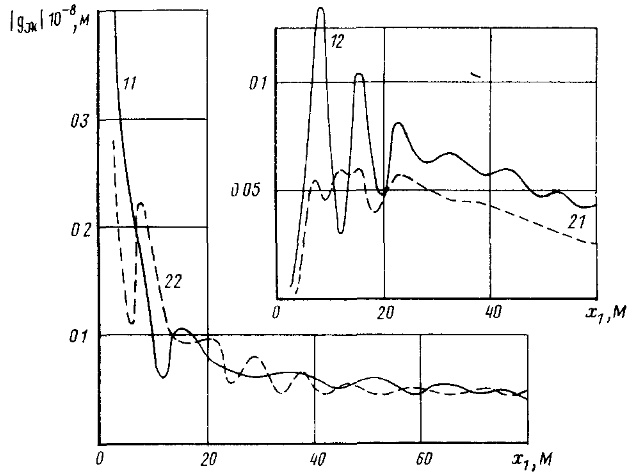

Fig.1

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